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Exploring the vicinity of the Bogomol'nyi–Prasad–Sommerfield bound

C A G Almeida, D Bazeia and L Losano

Departamento de Física, Universidade Federal da Paraíba, Caixa Postal 5008, 58051-970 João Pessoa, Paraíba, Brazil

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Abstract

We investigate systems of real scalar fields in bidimensional spacetime, dealing with potentials that are small modifications of potentials that admit supersymmetric extensions. The modifications are controlled by a real parameter, which allows the implementation of a perturbation procedure when such a parameter is small. The procedure allows one to obtain the energy and topological charge in closed forms, up to first order in the parameter. We illustrate the procedure with some examples. In particular, we show how to remove the degeneracy in energy for the one-field and the two-field solutions that appear in a model of two real scalar fields.

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1. Introduction

Domain walls are defect structures that appear in systems engendering spontaneous breaking of discrete symmetry. They are of interest for instance in condensed matter, as interfaces in magnetic materials [1], as seeds for pattern formation [2], and as interfaces in ferroelectric crystals [3–5], and in cosmology, as seeds for the formation of structures [6, 7] in the early universe. Domain walls spring from the immersion of kink-like solutions of (1, 1) spacetime dimensions to higher spatial dimensions. In general, kinks or domain walls appear in scalar field models, but they may also be present in extended systems, which include fermionic fields, with or without supersymmetry.

Recently, the study of supersymmetric models has brought new issues, such as for instance in the investigations of a Wess–Zumino model [8,9] engendering the Z_3 symmetry. This model allows the identification of a Bogomol'nyi equation [10] for a triple junction that breaks $\frac{1}{4}$ supersymmetry of the model. We learn from these works that supersymmetry helps easing calculations concerning the presence of the triple junction [11, 12]. However, supersymmetry seems to play no central role when the issue is the tiling of the plane and not the triple junction itself [13–15]. Furthermore, in [16] it was shown how one could entrap a planar regular hexagonal network of defects inside a domain wall. This model involves three real scalar fields, and engenders the $Z_2 \times Z_3$ symmetry, but it seems to have no supersymmetric extension.

These facts motivate the study of bosonic models that do not support supersymmetric extensions. However, to keep track of supersymmetry we investigate models that are close to the real bosonic portions of supersymmetric systems. These models are defined by potentials that contain two parts: the first part defines the real bosonic portion of a supersymmetric system; the second part defines the extended model. The first part alone constitutes the basic model, which supports static field configurations that minimize the energy, attained by static field configurations that solve first-order differential equations. In particular, we examine models that support topological solutions that belong to the same topological sector, and are degenerate, having the very same energy. An example of this was investigated in [17], in a model of two coupled scalar fields. This model has recently been extended to the case of several scalar fields in [18]. We explore the possibility of extending this system, in order to remove the degeneracy between the one-field and the two-field solutions. This investigation is of intrinsic interest, and may also help in examining applications to cosmology and to condensed matter. In cosmology we recall the usual route [6, 7], and also the new possibility [16]. In condensed matter, we envisage other issues, concerning for instance the presence of Ising and Bloch wall interfaces in magnetic materials described by the anisotropic *XY* model [2, 16], and the structural phase transition in ferroelectric crystals [5, 19].

The real bosonic sector of supersymmetric systems described by n chiral superfields $\Phi_c^1, \Phi_c^2, \ldots, \Phi_c^n$ contains n real scalar fields $\phi_1, \phi_2, \ldots, \phi_n$. The potential V is written in terms of the superpotential W, in such a way that for $V = V(\phi_1, \phi_2, \ldots, \phi_n)$ and $W = W(\phi_1, \phi_2, \ldots, \phi_n)$ we obtain

$$V = \tfrac{1}{2} W_{\phi_1}^2 + \tfrac{1}{2} W_{\phi_2}^2 + \cdots + \tfrac{1}{2} W_{\phi_n}^2 \tag{1}$$

where $W_{\phi_i} = \partial W/\partial\phi_i, i = 1, 2, \ldots, n$. These systems have been investigated in several different contexts in [15, 17, 19], and in references therein. In this paper we investigate systems where the potential includes an extra term, that modifies the above potential according to

$$V_\epsilon = V + \tfrac{1}{2}\epsilon F \tag{2}$$

where ϵ is a real parameter, and $F = F(\phi_1, \phi_2, \ldots, \phi_n)$ is in principle an arbitrary function of the fields. In this paper we show that if ϵ is small, we can develop a perturbation procedure that gives closed results up to first order in ϵ. The perturbation procedure is based on previous investigations [20, 21], and works nicely for potentials of the form (2) when the function F obeys some restrictions. We start our investigations using natural units, but we work with dimensionless fields and coordinates, and sometimes we refer to the model with $\epsilon = 0$ as the primary system, and to the complete model, with $\epsilon \neq 0$, as the extended system.

2. General considerations

In order to better understand the procedure, let us first consider models that are bosonic portions of supersymmetric theories. In this case we have

$$\mathcal{L} = \frac{1}{2} \sum_{i=1}^{n} \partial_\alpha\phi_i \partial^\alpha\phi_i - V(\phi_1, \ldots, \phi_n) \tag{3}$$

where V is given by equation (1). We work in (1, 1) spacetime dimensions. The equations of motion for static fields are

$$\frac{d^2\phi_1}{dx^2} = W_{\phi_1} W_{\phi_1\phi_1} + \cdots + W_{\phi_n} W_{\phi_n\phi_1}$$
$$\frac{d^2\phi_2}{dx^2} = W_{\phi_1} W_{\phi_1\phi_2} + \cdots + W_{\phi_n} W_{\phi_n\phi_2}$$
$$\vdots$$
$$\frac{d^2\phi_n}{dx^2} = W_{\phi_1} W_{\phi_1\phi_n} + \cdots + W_{\phi_n} W_{\phi_n\phi_n}.$$

These second-order equations are solved by field configurations that solve the first-order differential equations

$$\frac{d\phi_1}{dx} = W_{\phi_1} \qquad \frac{d\phi_2}{dx} = W_{\phi_2} \qquad \ldots \qquad \frac{d\phi_n}{dx} = W_{\phi_n}.$$

These are the Bogomol'nyi equations. The energy of the static solutions can be written as

$$E = \frac{1}{2}\int_{-\infty}^{\infty} dx \sum_{i=1}^{n}\left[\left(\frac{d\phi_i}{dx}\right)^2 + W_{\phi_i}^2\right]. \tag{4}$$

In general, the system may have several distinct sectors, which may be identified by the two vacuum states the static solution connect. Thus, if we use the set of numbers $\{v_a, v_b, v_c, \ldots\}$ to mark the vacuum states of the model, we can write

$$E^{ab} = E_B^{ab} + \frac{1}{2}\int_{-\infty}^{\infty} dx \sum_{i=1}^{n}\left(\frac{d\phi_i}{dx} - W_{\phi_i}\right)^2. \tag{5}$$

Here $E_B^{ab} = |W_{ab}|$ and $W_{ab} = W(v_a) - W(v_b)$. A pair of vacua defines a topological sector, and in the sector (ab) the energy is minimized to the bound E_B^{ab} for field configurations that obey the above first-order equations. Solutions of the Bogomol'nyi equations are called Bogomol'nyi–Prasad–Sommerfield solutions [10, 22], and we refer to this bound as the BPS bound. It is possible that $W(v_a) = W(v_b)$, giving a vanishing W_{ab}. In this case the topological sector cannot support BPS states, and we refer to this as a non-BPS sector (see [15]). The topological features of these solutions can be accounted for by introducing, for instance, the topological current [14]

$$j^\alpha = \tfrac{1}{2}\varepsilon^{\alpha\beta}\partial_\beta\Phi \tag{6}$$

where Φ is a column vector, such that $\Phi^t = (\phi_1\, \phi_2 \cdots \phi_n)$.

We illustrate the general situation with some examples. We work with natural units. However, we avoid unimportant considerations by considering the Lagrangian density $\mathcal{L}$, written in terms of dimensionless fields and coordinates. The usual form of the Lagrangian density is $\mathcal{L}' = \gamma\mathcal{L}$, where γ is a constant, which carries the correct dimension of the Lagrangian density in natural units.

2.1. One real scalar field

In the case of one real scalar field we have

$$\mathcal{L} = \tfrac{1}{2}\partial_\alpha\phi\partial^\alpha\phi - V(\phi). \tag{7}$$

The potential $V(\phi)$ specifies the system. As an example we consider the model

$$V(\phi) = \tfrac{1}{2}\phi^2 - |\phi| + \tfrac{1}{2}. \tag{8}$$

This potential was recently considered in [23]. There are two minima, at the values $\{v_1 = 1, v_2 = -1\}$. The equation of motion is $\mathrm{d}^2\phi/\mathrm{d}x^2 = \phi - \phi/|\phi|$. It has the solutions $\phi_\pm(x) = \pm(x/|x|)[1 - \exp(-|x|)]$. This model can be written in terms of the function

$$W(\phi) = \phi - \tfrac{1}{2}|\phi|\phi. \tag{9}$$

This means that the topological sector is a BPS sector. The BPS solution satisfies

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = 1 - |\phi|. \tag{10}$$

This solution is $\phi(x) = (x/|x|)\,[1 - \exp(-|x|)]$. It has a kink-like profile, and is linearly stable [24], minimizing the energy to $E_B = 1$.

2.2. *Two real scalar fields*

We exemplify the case of two real scalar fields by considering the superpotential

$$W(\phi, \chi) = \phi - \tfrac{1}{3}\phi^3 - r\phi\chi^2 \tag{11}$$

where r is a real parameter. Here the potential becomes

$$V(\phi, \chi) = \tfrac{1}{2}(\phi^2 - 1)^2 - r\chi^2 + r(1 + 2r)\phi^2\chi^2 + \tfrac{1}{2}r^2\chi^4. \tag{12}$$

There are two minima for $r < 0$, $v_1 = (1, 0)$, $v_2 = (-1, 0)$. For $r > 0$ there are four minima, the two former ones, and also the other two $v_3 = (0, \sqrt{1/r})$, $v_4 = (0, -\sqrt{1/r})$.

In this model the first-order equations are

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = (1 - \phi^2) - r\chi^2 \tag{13}$$

$$\frac{\mathrm{d}\chi}{\mathrm{d}x} = -2r\phi\chi. \tag{14}$$

We see that for $\chi \to 0$ the first-order equations demand $\mathrm{d}\phi/\mathrm{d}x = (1 - \phi^2)$, and the BPS defect solution is $\phi(x) = \tanh(x)$. However, for $\phi \to 0$ the first-order equations demand that $\chi^2 = 1/r$, that is, that the χ field should be at the corresponding minima. These results are manifestations that the sector defined by the minima $(\pm 1, 0)$ is a BPS sector, while the other sector, defined by the minima $(0, \pm\sqrt{1/r})$ is a non-BPS sector, and the non-BPS solutions are stable if and only if $1/r < 1$ [15].

The solution $\phi = \tanh(x)$ and $\chi = 0$ is a one-field solution, but the system also supports two-field solutions. In the sector that connects the minima $(\pm 1, 0)$, the two-field BPS solutions have the explicit form

$$\phi(x) = \tanh(2rx) \tag{15}$$

$$\chi(x) = \pm\sqrt{\frac{1}{r} - 2}\,\mathrm{sech}(2rx) \tag{16}$$

which requires that $1/r > 2$. The one- and two-field solutions that appear in the sector connecting the minima $(\pm 1, 0)$ have the same energy, $E = \frac{4}{3}$. In the (ϕ, χ) space the real line $x \in \mathbb{R}$ is mapped to line segments: for the one-field solution we obtain a straight line segment going from $(-1, 0)$ to $(1, 0)$, and for the two-field solutions we obtain an elliptic arc, obeying $\phi^2 + \chi^2/(1/r - 2) = 1$. These solutions map the Ising and Bloch walls that appear in magnetic systems, respectively, and represent solutions of the anisotropic *XY* model, that describe interfaces between ferromagnetic domains [2, 16].

3. Extended systems

Here we deal with systems defined by extended potentials, that differ from the above primary potentials by some slight modifications. We start by dealing with the general model

$$V_\epsilon = \tfrac{1}{2}\sum_{i=1}^{n} W_{\phi_i}^2 + \tfrac{1}{2}\epsilon F(\phi_1, \ldots, \phi_n) \tag{17}$$

where ϵ is a real parameter, infinitesimal. In this case we can use such a parameter to control the procedure of extending the result beyond the ϵ-independent value. The correction to the potential depends on $F = F(\phi_1, \ldots, \phi_n)$. In general, we may have two different types of functions: functions that respect and functions that do not respect the symmetry of the original system. Both type of functions are important to describe situations where the system is modified by the presence of external fields, chemical potentials, etc. However, in this paper we are interested mainly in topological solitons, in investigating the topological sectors of the model. Thus, we consider the case of functions that respect some symmetry of the primary system. In this case the function F accounts for modifications of the original system, without destroying the topological sector one is investigating. This means that in the set of possible vacuum states $\{v_a, v_b, v_c, \ldots\}$ of the primary system, at least the vacua v_a and v_b remain present, although they can be slightly changed to v_a^ϵ and v_b^ϵ. We shall be investigating slight modifications in the topological sectors of the BPS type, modifications that do not destroy the sectors themselves. In this case we write the static solution $\phi_1^\epsilon(x), \phi_2^\epsilon(x), \ldots, \phi_n^\epsilon(x)$ of the extended system in terms of the static solution of the original model in the form, up to first order in ϵ

$$\phi_i^\epsilon(x) = \phi_i(x) + \epsilon\, \eta_i(x) \qquad i = 1, 2, \ldots, n. \tag{18}$$

In the extended system, the energy of static solutions has the form

$$E = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \sum_{i=1}^{n}\left[\left(\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\right)^2 + W_{\phi_i}^2\right] + \frac{1}{2}\epsilon \int_{-\infty}^{\infty} \mathrm{d}x\, F(\phi_1, \phi_2, \ldots, \phi_n). \tag{19}$$

It can be written as, in the case of the BPS sector that connects the vacuum states labelled by a and b,

$$E^{ab} = \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \sum_{i=1}^{n}\left(\frac{\mathrm{d}\phi_i}{\mathrm{d}x} - W_{\phi_i}\right)^2 + E_B^{ab} + \frac{1}{2}\epsilon \int_{-\infty}^{\infty} \mathrm{d}x\, F(\phi_1, \phi_2, \ldots, \phi_n). \tag{20}$$

We now follow [20]. We see that the field ϕ_i^ϵ given by equation (18) shows that the first term in the above expression for the energy do not contribute to first order in ϵ. This fact allows us to write the energy in the form

$$E^{ab} = E_B^{ab} + \tfrac{1}{2}\epsilon \int_{-\infty}^{\infty} \mathrm{d}x\, F(\phi_1, \phi_2, \ldots, \phi_n) + \mathcal{O}(\epsilon^2). \tag{21}$$

Here $E_B^{ab} = |W_{ab}|$ and $W_{ab} = W(v_a^\epsilon) - W(v_b^\epsilon)$. The correction to the potential is small, and we separate two cases: first, the case where the correction does not change the minima of the primary potential, that is, the case where $v_a^\epsilon = v_a$ and $v_b^\epsilon = v_b$; and second, the case where the correction slightly modifies the minima of the primary potential. In the first case both $\phi_i^\epsilon(x)$ and $\phi_i(x)$ have the same asymptotic behaviour, thus $\eta_i(x)$ must vanish asymptotically; in the second case $\phi_i^\epsilon(x)$ is asymptotically different from $\phi_i(x)$, thus $\eta_i(x)$ cannot vanish asymptotically. In both cases the term E_B^{ab} exactly reproduces the corresponding term in the primary system, up to first order in ϵ. This is so because $W[\phi_1^\epsilon(\pm\infty),\ldots,\phi_n^\epsilon(\pm\infty)]$ can be expanded to give

$$W[\phi_1(\pm\infty),\ldots,\phi_n(\pm\infty)]+\epsilon\sum_{i=1}^n\eta_i\,(\pm\infty)\left.\frac{\mathrm{d}W}{\mathrm{d}\phi_i}\right|_{\phi_i(\pm\infty)}.$$

However, we know that $\phi_i(\pm\infty)$ are minima of the primary model, and are extrema of W. Thus, the second term in the above expression vanishes. For the topological charge, in the first case we see that it does not change, giving $Q_T^\epsilon = Q_T$. However, in the second case the topological charges change according to $Q_T^\epsilon = Q_T + \epsilon\,\Delta Q$, where Q_T^ϵ, Q_T and ΔQ are n-component vectors, and ΔQ accounts for the difference $\eta_i(\infty)-\eta_i(-\infty)$, $i=1,2,\ldots,n$.

The above results are general results, and we illustrate the general procedure with some examples, splitting the investigation into the two subsections that follow, which deal with one and two real scalar fields separately.

3.1. The case of one real scalar field

In the case of one field, let us first consider the ϕ^4 model, defined by the potential

$$V(\phi)=\tfrac{1}{2}(\phi^2-1)^2. \tag{22}$$

We consider $F_1(\phi)=(\phi^2-1)^2$, which is an example where the correction does not change the minima of the primary system, and so $Q_T^\epsilon = Q_T = 1$. In this case the energy becomes

$$E=E_B\left(1+\tfrac{1}{2}\epsilon\right) \tag{23}$$

where $E_B=\frac{4}{3}$. It can be greater ($\epsilon>0$) or less ($\epsilon<0$) than the energy of the unperturbed system.

We note that the above $F_1(\phi)$ allows us to rewrite the potential as

$$V(\phi)=\tfrac{1}{2}(1+\epsilon)(\phi^2-1)^2. \tag{24}$$

This potential requires that $\epsilon>-1$, and shows that the extended system is very much like the primary model, with the coupling for self-interaction changed by the ϵ term. In this case we can find the energy of the static solution exactly. It is

$$E=E_B\sqrt{1+\epsilon}. \tag{25}$$

For ϵ very small, we expand the above result to obtain the former answer, equation (23), and this shows that our approach works correctly.

As a second example, let us consider the same primary model and another function

$$F_2(\phi)=(1-\phi^2). \tag{26}$$

In this case the correction does change the minima of the primary potential. The new minima are at $\pm(1+\epsilon/4)$. The energy is

$$E=E_B\left(1+\tfrac{3}{4}\epsilon\right). \tag{27}$$

It can be greater ($\epsilon>0$) or less ($\epsilon<0$) than the energy of the unperturbed system. The topological charge changes to $Q_T^\epsilon = Q_T\,(1+\epsilon/4)$.

Here we also note that with the above correction of equation (26) the potential can be rewritten in the form

$$V(\phi)=\tfrac{1}{2}\left(\phi^2-1-\tfrac{1}{2}\epsilon\right)^2 \tag{28}$$

which is correct to first order in ϵ. In this case the energy of the static solution is

$$E = E_B \left(1 + \tfrac{1}{2}\epsilon\right)^{3/2}. \tag{29}$$

However, since ϵ is very small, we expand the above result to obtain the former answer, equation (27). We note that the modification in the minima of the potential changes the topological charge, and also the energy of the topological solution. The energy increases or decreases, depending on the increasing or decreasing of the spontaneous symmetry breaking parameter.

We now consider the model defined in equation (8). We extend this model with the above $F_2(\phi)$. The new potential is

$$V_\epsilon(\phi) = \tfrac{1}{2}\phi^2 - |\phi| + \tfrac{1}{2} + \tfrac{1}{2}\epsilon(1-\phi^2). \tag{30}$$

For $\epsilon \neq 0$, small, the minima change from $v_\pm = \pm 1$ to $v_\pm^\epsilon = \pm(1+\epsilon)$. The energy of the topological solution changes to

$$E = 1 + \tfrac{3}{2}\epsilon. \tag{31}$$

We see that the energy decreases when the spontaneous symmetry breaking parameter decreases.

We note that the above potential can be written as

$$V(\phi) = \frac{1}{2}\left(\sqrt{1+\epsilon} - \frac{|\phi|}{\sqrt{1+\epsilon}}\right)^2. \tag{32}$$

This result is valid up to first order in ϵ. It allows us to introduce a superpotential, and the first-order equation is

$$\frac{\mathrm{d}\phi}{\mathrm{d}x} = \sqrt{1+\epsilon} - \frac{|\phi|}{\sqrt{1+\epsilon}}. \tag{33}$$

It has the BPS solution

$$\phi(x) = (1+\epsilon)\frac{x}{|x|}\left(1 - \mathrm{e}^{-|x|/(1+\epsilon)}\right). \tag{34}$$

This shows that both the amplitude and width of the former kink-like solution change in the extended model. The energy is $(1+\epsilon)^{3/2}$, but since ϵ is small we can expand this result to obtain the former answer, given by equation (31).

3.2. The case of two real scalar fields

The case of two fields is more involved, and we envisage several distinct possibilities of illustrating this situation. We consider the example presented in section 2. We extend that model with the function

$$F_1(\phi, \chi) = (1-\phi^2). \tag{35}$$

In this case, in the BPS sector defined by the minima $(\pm 1, 0)$ the correction to the one-field solution is

$$E_1^1 = E_B\left(1 + \tfrac{3}{4}\epsilon\right). \tag{36}$$

In the case of the two-field solution (15) and (16) we obtain

$$E_2^1 = E_B\left(1 + \frac{3}{8}\,\epsilon\,\frac{1}{r}\right). \tag{37}$$

We introduce the ratio between energies, $R = E_2/E_1$. We see that

$$R_1 = 1 + \frac{3}{8}\epsilon\left(\frac{1}{r} - 2\right). \tag{38}$$

We consider another perturbation

$$F_2(\phi, \chi) = r\chi^2. \tag{39}$$

It gives

$$E_1^2 = E_B \tag{40}$$

and

$$E_2^2 = E_B\left[1 + \frac{3}{8}\epsilon\left(\frac{1}{r} - 2\right)\right]. \tag{41}$$

They give

$$R_2 = 1 + \frac{3}{8}\epsilon\left(\frac{1}{r} - 2\right). \tag{42}$$

We see that $R_1 = R_2$. The ratio $R = E_2/E_1$ does not depend on the way one extends the model, using $F_1 = (1-\phi^2)$ or $F_2 = r\chi^2$. We note that the extension with $F_1 = (1-\phi^2)$ changes the minima of the primary model from $(\pm1, 0)$ to $(\pm1 \pm \epsilon/4), 0)$. Thus, the topological charge of the sector changes according to

$$Q_T = \begin{pmatrix}1\\0\end{pmatrix} \rightarrow Q_T^\epsilon = Q_T + \epsilon\begin{pmatrix}\frac{1}{4}\\0\end{pmatrix}. \tag{43}$$

The other extension, that uses $F_2 = r\chi^2$, does not change the minima $(\pm1, 0)$, so the topological charge in this BPS sector remains the same $Q_T^\epsilon = Q_T$. The above examples show two distinct ways of removing the degeneracy between the standard one- and two-field solutions (15) and (16). The two-field solutions are less energetic for $\epsilon < 0$. This means that one favours the non-trivial two-field configuration when the symmetry breaking parameter decreases in one or in the two ϕ and χ directions.

There is another way of removing the degeneracy of the one- and two-field solutions that appear in the sector connecting the minima $(\pm1, 0)$. We consider the case where the extra part contains interactions between the two fields, for instance

$$F_3^{(k)}(\phi, \chi) = r\phi^{2k}\chi^2 \qquad k = 1, 2, \ldots. \tag{44}$$

In this case the minima of the primary system do not change, so the topological charge is not modified. The same happens to the energy of the one-field solution. However, the energy of the two-field solutions changes to

$$E^{(k)} = E_B + \frac{1}{2}\epsilon\left(\frac{1}{r} - 2\right)\frac{1}{2k+1}. \tag{45}$$

We see that in the limit $k \rightarrow 0$ one obtains the former result, given by equation (41). We also see that the sign of ϵ makes the energy density of the two-field solutions higher ($\epsilon > 0$) or lower ($\epsilon < 0$) than the energy density of the one-field solution, removing the degeneracy they have in the primary system.

In the former model of two real scalar fields with the function $F_1(\phi, \chi)$ as in equation (35), the potential has the form

$$V_\epsilon(\phi, \chi) = \tfrac{1}{2}(\phi^2 - 1)^2 - r\chi^2 + \tfrac{1}{2}r^2\chi^4 + r(1+2r)\phi^2\chi^2 + \tfrac{1}{2}\epsilon(1-\phi^2). \tag{46}$$

There are four minima, two at $(0, \chi_\pm)$, $\chi_\pm = \pm\sqrt{1/r}$, and two at $(\phi_\pm, 0)$, $\phi_\pm = \pm(1+\epsilon/4)$. We compare this potential with the potential of the primary model. The presence of the extra term shows that, for $r > 1$ the stable non-BPS solution implies

$$V_\epsilon(\phi, 0) = \tfrac{1}{2}(\phi^2 - 1)^2 + \tfrac{1}{2}\epsilon(1 - \phi^2) \tag{47}$$

and

$$V_\epsilon(\phi, \pm\sqrt{1/r}) = \tfrac{1}{2}\phi^4 + \left(2r - \tfrac{1}{2}\epsilon\right)\phi^2 + \tfrac{1}{2}\epsilon. \tag{48}$$

This shows that the (squared) mass of the ϕ-meson is, inside the χ-kink

$$m_\phi^2(\text{in}) = 4\left(1 + \tfrac{1}{2}\epsilon\right). \tag{49}$$

Outside the χ-kink we obtain $m_\phi^2(\text{out}) = 4r - \epsilon$, which gives the ratio

$$\frac{m_\phi^2(\text{in})}{m_\phi^2(\text{out})} = \frac{1}{r}\left[1 + \frac{1}{2}\left(1 + \frac{1}{2r}\right)\epsilon\right]. \tag{50}$$

Thus, if the non-BPS χ-kink entraps ϕ-mesons in the primary system, the entrapment is still more efficient in the extended system, for $\epsilon < 0$. We see that deviations from the BPS bound may improve the efficiency of the mechanism for the entrapment of the other field.

3.3. Another case

Let us now consider the model investigated in [16]. It is described by three real scalar fields, and the potential has the form

$$V(\sigma, \phi, \chi) = \tfrac{2}{3}\left(\sigma^2 - \tfrac{9}{4}\right)^2 + \left(r\sigma^2 - \tfrac{9}{4}\right)(\phi^2 + \chi^2) + (\phi^2 + \chi^2)^2 - \phi(\phi^2 - 3\chi^2). \tag{51}$$

The projection with $(\phi, \chi) \to (0, 0)$ gives

$$V(\sigma) = \tfrac{2}{3}\left(\sigma^2 - \tfrac{9}{4}\right)^2. \tag{52}$$

This potential can be written with the superpotential $W(\sigma) = (3\sqrt{3}/2)\sigma - (2\sqrt{3}/9)\sigma^3$. This fact shows that the defect $\sigma(x) = \frac{3}{2}\tanh(\sqrt{3}x)$ that appears in this case is a BPS defect. We can then extend this model adding to the potential in equation (51) a term depending on the σ field, for instance

$$f(\sigma) = \tfrac{1}{2}\epsilon\left(\tfrac{3}{2} - \sigma^2\right). \tag{53}$$

We use former results to see that this term contributes to decreasing or increasing the energy of the basic defect, increasing or decreasing the efficiency of the mechanism for the entrapment of the network the other two fields ϕ and χ may generate.

4. Comments

In this paper we have examined systems described by real scalar fields, in which the energy of static field configurations is in the vicinity of the BPS bound. This bound is attained by field configurations that solve first-order equations, and minimize the energy. The BPS bound appears in the real bosonic sector of supersymmetric theories described by chiral superfields. The systems we have investigated are extensions of primary systems, described by potentials given by the functions $W = W(\phi_1, \phi_2, \ldots)$ and $F = F(\phi_1, \phi_2, \ldots)$, in the specific form

$$\tfrac{1}{2}W_{\phi_1}^2 + \tfrac{1}{2}W_{\phi_2}^2 + \cdots + \tfrac{1}{2}W_{\phi_n}^2 + \tfrac{1}{2}\epsilon\, F(\phi_1, \phi_2, \ldots, \phi_n).$$

The parameter ϵ is real, and is used to control the deviation from the primary system, described in terms of the superpotential W.

In the extended system, if the static solutions are similar to the solutions one finds in the primary system, that is, for $\phi_i^\epsilon(x) = \phi_i(x) + \epsilon\, \eta_i(x)$, we could examine the energy related to the new defect solutions $\phi_i^\epsilon(x)$, and write the first corrections in ϵ in a closed form, independently of the specific form of the new defect solution itself. The formal results are of direct interest to field theory, where they may be used to improve the mechanism for the entrapment of the other field [25–30].

We have investigated specific systems, and we have found diverse possibilities of removing the degeneracy between different types of solutions, without destroying the degeneracy of the vacuum states. The examples we have presented serve to illustrate some practical possibilities of removing defect degeneration, and this is of direct interest in application in specific physical situations. In ferroelectric crystals, for instance, the order parameters that control structural phase transitions may be changed by the application of external pressure along specific planar directions in the crystal. This is a typical scenario for changing the parameters that control spontaneous symmetry breaking inside the crystalline material, changing the energetics of the structural phase transition. In the systems we have examined in this paper, the presence of external pressure may be directly mapped into specific forms of F that control the extended system. This paper opens a new route for exploring systems of coupled scalar fields, intending to mimic specific systems in applications to cosmology and to condensed matter. We postpone to the near future some specific investigations.

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